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Some new nonlinear integral inequalities and their applications in the qualitative analysis of differential equations

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Abstract

In this paper, some new nonlinear integral inequalities are established, which provide a handy tool for analyzing the global existence and boundedness of solutions of differential and integral equations. The established results generalize the main results in Sun (J. Math. Anal. Appl. **301**, 265-275, 2005), Ferreira and Torres (Appl. Math. Lett. **22**, 876-881, 2009), Xu and Sun (Appl. Math. Comput. 182, 1260-1266, 2006) and Li et al. (J. Math. Anal. Appl. **372**, 339-349 2010).

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1 Introduction

During the past decades, with the development of the theory of differential and integral equations, a lot of integral inequalities, for example [1-12], have been discovered, which play an important role in the research of boundedness, global existence, stability of solutions of differential and integral equations.

In [9], the following two theorems for retarded integral inequalities were established.

Theorem A: $R_+ = [0, \infty)$. Let u, f, g be nondecreasing continuous functions defined on R_+ and let c be a nonnegative constant. Moreover, let $\omega \in C(R_+, R_+)$ be nondecreasing with $\omega(u) > 0$ on $(0, \infty)$ and $\alpha \in C^1(R_+, R_+)$ be nondecreasing with $\alpha(t) \leq t$ on R_+. m, n are constants, and $m > n > 0$. If

$$u^m(t) \leq c^{\frac{m}{m-n}} + \frac{m}{m-n}\int_0^{\alpha(t)} [f(s)u^n(s)\omega(u(s)) + g(s)u^n(s)]\, ds, \quad t \in R_+,$$

then for $t \in [0, \xi]$

$$u(t) \leq \{\Omega^{-1}[\Omega(c + \int_0^{\alpha(t)} g(s)ds) + \int_0^{\alpha(t)} f(s)ds]\}^{\frac{1}{m-n}},$$

where $\Omega(r) = \int_1^r \frac{1}{\omega(s^{\frac{1}{m-n}})}ds$, $r > 0$, Ω^{-1} is the inverse of Ω, $\Omega(\infty) = \infty$, and $\xi \in R_+$ is chosen so that $\Omega(c + \int_0^{\alpha(t)} g(s)ds) + \int_0^{\alpha(t)} f(s)ds \in Dom(\Omega^{-1})$.

Theorem B: Under the hypothesis of Theorem B, if

$$u^m(t) \le c^{\frac{m}{m-n}} + \frac{m}{m-n}\int_0^{\alpha(t)} f(s)u^n(s)\omega(u(s))ds + \frac{m}{m-n}\int_0^t g(s)u^n(s)\omega(u(s))ds, \quad t \in R_+,$$

then for $t \in [0, \xi]$

$$u(t) \le \{\Omega^{-1}[\Omega(c) + \int_0^{\alpha(t)} f(s)ds + \int_0^t g(s)ds]\}^{\frac{1}{m-n}}.$$

Recently, in [10], the author provided a more general result.

Theorem C: $R_0^+ = [0, \infty)$, $R^+ = (0, \infty)$. Let $f(t, s)$ and $g(t, s) \in C(R_0^+ \times R_0^+, R_0^+)$ be nondecreasing in t for every s fixed. Moreover, let $\phi \in C(R_0^+, R_0^+)$ be a strictly increasing function such that $\lim_{x\to\infty} \phi(x) = \infty$ and suppose that $c \in C(R_0^+, R_+)$ is a nondecreasing function. Further, let $\eta, \omega \in C(R_0^+, R_0^+)$ be nondecreasing with $\{\eta, \omega\}(x) > 0$ for $x \in (0, \infty)$ and $\int_{x_0}^{\infty} \frac{1}{\eta(\phi^{-1}(s))} ds = \infty$, with x_0 defined as below. Finally, assume that $\alpha \in C^1(R_0^+, R_0^+)$ is nondecreasing with $\alpha(t) \le t$. If $u \in C(R_0^+, R_0^+)$ satisfies

$$\phi(u(t)) \le c(t) + \int_0^{\alpha(t)} [f(t,s)\eta(u(s))\omega(u(s)) + g(t,s)\eta(u(s))]ds, \quad t \in R_0^+$$

then there exists $\tau \in R^+$ so that for all $t \in [0, \tau]$ we have

$$\psi(p(t)) + \int_0^{\alpha(t)} f(t,s)\, ds \in Dom\ (\psi^{-1}),$$

and

$$u(t) \le \phi^{-1}\{G^{-1}(\psi^{-1}[\psi(p(t)) + \int_0^{\alpha(t)} f(t,s)ds])\},$$

where $G(x) = \int_{x_0}^{x} \frac{1}{\eta(\phi^{-1}(s))} ds$.

with $x \ge c(0) > x_0 > 0$ if $\int_0^x \frac{1}{\eta(\phi^{-1}(s))} ds = \infty$ and $x \ge c(0) > x_0 \ge 0$ if

$$\int_0^x \frac{1}{\eta(\phi^{-1}(s))} ds < \infty.$$

$$p(t) = G(c(t)) + \int_0^{\alpha(t)} g(t,s)ds,$$

$$\psi(x) = \int_{x_1}^{x} \frac{1}{\omega(\phi^{-1}(G^{-1}(s)))}, \ x > 0, x_1 > 0.$$

Here G^{-1} and ψ^{-1} are inverse functions of G and ψ, respectively.

In [11], Xu presented the following two theorems:

Theorem D: $R_+ = [0, \infty)$. Let u, f, g be real-valued nonnegative continuous functions defined for $x \ge 0$, $y \ge 0$ and let c be a nonnegative constant. Moreover, let $\omega \in C(R_+, R_+)$ be nondecreasing with $\omega(u) > 0$ on $(0, \infty)$ and $\alpha, \beta, \in C^1(R_+, R_+)$ be nondecreasing with $\alpha(x) \le x$, $\beta(y) \le y$ on R_+. m, n are constants, and $m > n > 0$. If

$$\begin{aligned} u^m(x,y) \le{} & a(x) + b(y) + \frac{m}{m-n}\int_0^{\alpha(x)}\int_0^{\beta(y)} [f(t,s)u^n(t,s)\omega(u(t,s)) \\ & + g(t,s)u^n(t,s)]\, dsdt, \quad x, y \in R_+, \end{aligned}$$

then for $x \in [0, \xi]$, $y \in [0, \eta]$

$$u(x,y) \leq \{\Omega^{-1}[\Omega(p(x,y)) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(t,s)dsdt]\}^{\frac{1}{m-n}},$$

where

$$p(x,y) =[a(0)+b(y)]^{\frac{m-n}{m}} + \frac{m-n}{m} \int_0^x \frac{a'(t)}{[a(t)+b(0)]^{\frac{n}{m}}}\, dt + \int_0^{\alpha(x)} \int_0^{\beta(y)} g(t,s)\, dsdt,$$

Ω is defined as in Theorem A, and ξ, η are chosen so that

$\Omega(p(x,y)) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(t,s)\, dsdt \in Dom(\Omega^{-1})$.

Theorem E: Under the hypothesis of Theorem D, if

$$u^m(x,y) \leq a(x)+b(y) + \frac{m}{m-n} \int_0^{\alpha(x)} \int_0^{\beta(y)} f(t,s)u^n(t,s)\omega(u(t,s))dsdt + \frac{m}{m-n} \int_0^x \int_0^y g(t,s)u^n(t,s)\omega(u(t,s))dsdt, \quad x,y \in R_+,$$

then

$$u(x,y) \leq \{\Omega^{-1}[\Omega(q(x,y)) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(t,s)dsdt + \int_0^x \int_0^y g(t,s)dsdt]\}^{\frac{1}{m-n}}$$

where

$$q(x,y) = [a(0)+b(y)]^{\frac{m-n}{m}} + \frac{m-n}{m} \int_0^x \frac{a'(t)}{[a(t)+b(0)]^{\frac{n}{m}}}\, dt.$$

In this paper, motivated by the above work, we will prove more general theorems and establish some new integral inequalities. Also we will give some examples so as to illustrate the validity of the present integral inequalities.

2 Main results

In the rest of the paper we denote the set of real numbers as R, and $R_+ = [0, \infty)$ is a subset of R. $Dom(f)$ and $Im(f)$ denote the definition domain and the image of f, respectively.

Theorem 2.1: Assume that x, $a \in C(R_+, R_+)$ and $a(t)$ is nondecreasing. f_i, g_i, h_i, $\partial_t f_i$, $\partial_t g_i$, $\partial_t h_i \in C(R_+ \times R_+, R_+)$, $i = 1, 2$. Let $\omega \in C(R_+, R_+)$ be nondecreasing with $\omega(u) > 0$ on $(0, \infty)$. p, q are constants, and $p > q > 0$. If $\alpha \in C^1(R_+, R_+)$ is nondecreasing with $\alpha(t) \leq t$ on R_+, and

$$x^p(t) \leq a(t) + \int_0^{\alpha(t)} [f_1(s,t)x^q(s)\omega(x(s)) + g_1(s,t)x^q(s) + \int_0^s h_1(\tau,t)x^q(\tau)d\tau]\, ds + \int_0^t [f_2(s,t)x^q(s)\omega(x(s)) + g_2(s,t)x^q(s) + \int_0^s h_2(\tau,t)x^q(\tau)d\tau]ds, \quad t \in R_+, \tag{1}$$

then there exists $\bar{t} \in R_+$ such that for $t \in [0, \bar{t}]$

$$x(t) \leq \{\Omega^{-1}\{\Omega(H(t)) + \frac{p-q}{p}[\int_0^{\alpha(t)} f_1(s,t) + \int_0^t f_2(s,t)ds]\}\}^{\frac{1}{p-q}}, \tag{2}$$

where $H(t) = a^{\frac{p-q}{p}}(t) + \frac{p-q}{p}\{\int_0^{\alpha(t)}[g_1(s,t) + \int_0^s h_1(\tau,t)d\tau]\,ds$

$$+\int_0^t [g_2(s,t) + \int_0^s h_2(\tau,t)d\tau]\,ds\}, \tag{3}$$

$\Omega(r) = \int_1^r \frac{1}{\omega(s^{\frac{1}{p-q}})}ds$, $r > 0$. Ω^{-1} is the inverse of Ω, and $\Omega(\infty) = \infty$.

Proof: The proof for the existence of $\bar{t}$ can be referred to Remark 1 in [10]. We notice (3) obviously holds for $t = 0$. Now given an arbitrary number $T \in (0, \bar{t}]$, for $t \in (0, T]$, we have

$$\begin{aligned} x^p(t) \le{}& a(T) + \int_0^{\alpha(t)} [f_1(s,t)x^q(s)\omega(x(s)) + g_1(s,t)x^q(s) + \int_0^s h_1(\tau,t)x^q(\tau)d\tau]\,ds \\ &+ \int_0^t [f_2(s,t)x^q(s)\omega(x(s)) + g_2(s,t)x^q(s) + \int_0^s h_2(\tau,t)x^q(\tau)d\tau]\,ds. \end{aligned} \tag{4}$$

Let the right-hand side of (4) be $z(t)$, then $x^p(t) \le z(t)$ and $x^p(\alpha(t)) \le z(\alpha(t)) \le z(t)$. So

$$\begin{aligned} z'(t) ={}& [f_1(\alpha(t),t)x^q(\alpha(t))\omega(x(\alpha(t))) + g_1(\alpha(t),t)x^q(\alpha(t)) + \int_0^{\alpha(t)} h_1(\tau,t)x^q(\tau)d\tau]\alpha'(t) \\ &+ \int_0^{\alpha(t)} [\frac{\partial f_1(s,t)}{\partial t}x^q(s)\omega(x(s)) + \frac{\partial g_1(s,t)}{\partial t}x^q(s) + \frac{\partial \int_0^s h_1(\tau,t)x^q(\tau)d\tau}{\partial t}]\,ds \\ &+ [f_2(t,t)x^q(t)\omega(x(t)) + g_2(t,t)x^q(t) + \int_0^t h_2(\tau,t)x^q(\tau)d\tau] \\ &+ \int_0^t [\frac{\partial f_2(s,t)}{\partial t}x^q(s)\omega(x(s)) + \frac{\partial g_2(s,t)}{\partial t}x^q(s) + \frac{\partial \int_0^s h_2(\tau,t)x^q(\tau)d\tau}{\partial t}]\,ds \\ \le{}& \{[f_1(\alpha(t),t)\omega(x(\alpha(t))) + g_1(\alpha(t),t) + \int_0^{\alpha(t)} h_1(\tau,t)d\tau]\alpha'(t) \\ &+ \int_0^{\alpha(t)} [\frac{\partial f_1(s,t)}{\partial t}\omega(x(s)) + \frac{\partial g_1(s,t)}{\partial t} + \frac{\partial \int_0^s h_1(\tau,t)d\tau}{\partial t}]\,ds \\ &+ [f_2(t,t)\omega(x(t)) + g_2(t,t) + \int_0^t h_2(\tau,t)d\tau] \\ &+ \int_0^t [\frac{\partial f_2(s,t)}{\partial t}\omega(x(s)) + \frac{\partial g_2(s,t)}{\partial t} + \frac{\partial \int_0^s h_2(\tau,t)d\tau}{\partial t}]ds\}z^{\frac{q}{p}}(t) \end{aligned}$$

Then

$$\begin{aligned} \frac{z'(t)}{z^{\frac{q}{p}}(t)} \le{}& \frac{d\int_0^{\alpha(t)}[f_1(s,t)\omega(x(s)) + g_1(s,t) + \int_0^s h_1(\tau,t)d\tau]\,ds}{dt} \\ &+ \frac{d\int_0^t [f_2(s,t)\omega(x(s)) + g_2(s,t) + \int_0^s h_2(\tau,t)d\tau]\,ds}{dt}. \end{aligned} \tag{5}$$

An integration for (5) from 0 to t, considering $z(0) = a(T)$, yields

$$\begin{aligned} z^{\frac{p-q}{p}}(t) \le{}& a^{\frac{p-q}{p}}(T) + \frac{p-q}{p}\{\int_0^{\alpha(t)} [f_1(s,t)\omega(x(s)) + g_1(s,t) + \int_0^s h_1(\tau,t)d\tau]\,ds \\ &+ \int_0^t [f_2(s,t)\omega(x(s)) + g_2(s,t) + \int_0^s h_2(\tau,t)d\tau]ds\}. \end{aligned} \tag{6}$$

Then

$$z^{\frac{p-q}{p}}(t) \le H(T) + \frac{p-q}{p}[\int_0^{\alpha(t)} f_1(s,t)\omega(z^{\frac{1}{p}}(s)) + \int_0^t f_2(s,t)\omega(z^{\frac{1}{p}}(s))ds]. \tag{7}$$

Let the right-hand side of (7) be $y(t)$. Then we have $z^{\frac{p-q}{p}}(t) \le y(t)$, $z^{\frac{p-q}{p}}(\alpha(t)) \le y(\alpha(t)) \le y(t)$, and

$$\begin{aligned} y'(t) &= \frac{p-q}{p}[f_1(\alpha(t),t)\omega(z^{\frac{1}{p}}(\alpha(t)))\alpha'(t) + \int_0^{\alpha(t)} \frac{\partial f_1(s,t)}{\partial t}\omega(z^{\frac{1}{p}}(s))\, ds \\ &\quad + f_2(t,t)\omega(z^{\frac{1}{p}}(t)) + \int_0^t \frac{\partial f_2(s,t)}{\partial t}\omega(z^{\frac{1}{p}}(s))ds] \\ &\le \frac{p-q}{p}\frac{d[\int_0^{\alpha(t)} f_1(s,t) + \int_0^t f_2(s,t)ds]}{dt}\omega(y^{\frac{1}{p-q}}(t)), \end{aligned} \tag{8}$$

that is

$$\frac{y'(t)}{\omega(y^{\frac{1}{p-q}}(t))} \le \frac{p-q}{p}\frac{d[\int_0^{\alpha(t)} f_1(s,t) + \int_0^t f_2(s,t)ds]}{dt}. \tag{9}$$

Integrating (9) from 0 to t, considering $y(0) = H(T)$, it follows

$$\Omega(y(t)) - \Omega(H(T)) \le \frac{p-q}{p}[\int_0^{\alpha(t)} f_1(s,t) + \int_0^t f_2(s,t)ds]. \tag{10}$$

So

$$\begin{aligned} &x(t) \le z^{\frac{1}{p}}(t) \le y^{\frac{1}{p-q}}(t) \le \\ &\{\Omega^{-1}\{\Omega(H(T)) + \frac{p-q}{p}[\int_0^{\alpha(t)} f_1(s,t) + \int_0^t f_2(s,t)ds]\}\}^{\frac{1}{p-q}}, t \in (0,\bar{t}]. \end{aligned} \tag{11}$$

Taking $t = T$ in (11), then

$$x(T) \le \{\Omega^{-1}\{\Omega(H(T)) + \frac{p-q}{p}[\int_0^{\alpha(T)} f_1(s,T) + \int_0^T f_2(s,T)ds]\}\}^{\frac{1}{p-q}},$$

Considering $T \in (0,\bar{t}]$ is arbitrary, substituting T with t, and then the proof is complete.

Remark 1 : We note that the right-hand side of (2) is well defined since $\Omega(\infty) = \infty$.

Remark 2 : If we take $p = 2$, $q = 1$, $\omega(u) = u$, $h_1(s,t) = h_2(s,t) \equiv 0$ or $p = 2$, $q = 1$, $h_1(s,t) = h_2(s,t) \equiv 0$, respectively, then our Theorem 2.1 reduces to [12, Theorems 2.1, 2.2].

Corollary 2.1: Assume that x, a, α, ω, Ω are defined as in Theorem 2.1. f_i, g_i, $h_i \in C(R_+, R_+)$, m_i, n_i, $l_i \in C^1(R_+, R_+)$, $i = 1, 2$. If

$$\begin{aligned} x^p(t) &\le a(t) + \int_0^{\alpha(t)} [m_1(t)f(s)x^q(s)\omega(x(s)) + n_1(t)g_1(s)x^q(s) \\ &\quad + \int_0^s l_1(t)h_1(\tau)x^q(\tau)d\tau]ds + \int_0^t [m_2(t)f_2(s)x^q(s)\omega(x(s)) + n_2(t)g_2(s)x^q(s) \\ &\quad + \int_0^s l_2(t)h_2(\tau)x^q(\tau)d\tau]ds, \quad t \in R_+, \end{aligned} \tag{12}$$

then we can find some $\bar{t} \in R_+$ such that for $t \in [0, \bar{t}]$

$$x(t) \le \{\Omega^{-1}\{\Omega(H(t)) + \frac{p-q}{p}[\int_0^{\alpha(t)} m_1(t) f_1(s)ds + \int_0^t m_2(t) f_2(s)ds]\}\}^{\frac{1}{p-q}}, \tag{13}$$

where

$$\begin{aligned} H(t) =& a^{\frac{p-q}{p}}(t) + \frac{p-q}{p}\{\int_0^{\alpha(t)} [n_1(t)g_1(s) + \int_0^s l_1(t)h_1(\tau)d\tau]\, ds \\ &+ \int_0^t [n_2(t)g_2(s) + \int_0^s l_2(t)h_2(\tau)d\tau]\, ds\}. \end{aligned} \tag{14}$$

Remark 3: If $a(t) \equiv C^{\frac{p}{p-q}}$, $m_1(t) = n_1(t) \equiv 1$, $l_1(t) \equiv 0$, $m_2(t) = n_2(t) = l_2(t) \equiv 0$ for $t \in R^+$, then Corollary 1 reduces to Theorem A [9, Theorem 2.1]. If $a(t) \equiv C^{\frac{p}{p-q}}$, $m_1(t) \equiv 1$, $g_1(t) \equiv 0$, $l_1(t) \equiv 0$, $m_2(t) \equiv 1$, $n_2(t) = l_2(t) \equiv 0$ for $t \in R^+$, then Corollary 2.1 reduces to Theorem B [9, Theorem 2.2].

Corollary 2:2: Assume that x, a, α, ω, Ω are defined as in Theorem 2.1. f, g, h, $\partial_t f$, $\partial_t g$, $\partial_t h \in C(R_+ \times R_+, R_+)$. If

$$x^p(t) \le a(t) + \int_0^{\alpha(t)} [f(s,t)x^q(s)\omega(x(s)) + g(s,t)x^q(s) + \int_0^s h(\tau,t)x^q(\tau)d\tau]\, ds, t \in R_+, \tag{15}$$

then for $t \in [0, \bar{t}]$

$$x(t) \le \{\Omega^{-1}[\Omega(H(t)) + \frac{p-q}{p}\int_0^{\alpha(t)} f(s,t)ds]\}^{\frac{1}{p-q}}, \tag{16}$$

where

$$H(t) = a^{\frac{p-q}{p}}(t) + \frac{p-q}{p}\{\int_0^{\alpha(t)} [g(s,t) + \int_0^s h(\tau,t)d\tau]\, ds\}. \tag{17}$$

Corollary 2:3: Assume that x, a, α, ω, Ω are defined as in Theorem 2.1. f, g, $h \in C(R_+, R_+)$, m, n, $l \in C^1(R_+, R_+)$. If

$$\begin{aligned} x^p(t) \le& a(t) + \int_0^{\alpha(t)} [m(t)f(s)x^q(s)\omega(x(s)) + n(t)g(s)x^q(s) \\ &+ \int_0^s l(t)h(\tau)x^q(\tau)d\tau]ds,\ t \in R_+, \end{aligned} \tag{18}$$

then for $t \in [0, \bar{t}]$

$$x(t) \le \{\Omega^{-1}[\Omega(H(t)) + \frac{p-q}{p}\int_0^{\alpha(t)} m(t)f(s)ds]\}^{\frac{1}{p-q}}, \tag{19}$$

where

$$H(t) = a^{\frac{p-q}{p}}(t) + \frac{p-q}{p}\{\int_0^{\alpha(t)} [n(t)g(s) + \int_0^s l(t)h(\tau)d\tau]\, ds\}. \tag{20}$$

Motivated by Corollary 2.2 and Theorem C [10], we will give the following more general theorem:

Theorem 2:2: Assume that $f(s, t)$, $g(s, t)$, $h(s, t) \in C(R_+ \times R_+, R_+)$ are nondecreasing in t for each s fixed, and $\varphi \in C(R_+, R_+)$ is a strictly increasing function with

$\lim_{x\to\infty}\phi(x)=\infty$. ψ, $\omega\in C(R_+, R_+)$ are nondecreasing with $\psi(x)>0$, $\omega(x)>0$ for $x\in(0,\infty)$ and $\int_{t_0}^{\infty}\frac{1}{\psi(\phi^{-1}(s))}ds=\infty$, $a(t)$, $\alpha(t)$ are defined as in Theorem 2.1, and $a(0)>t_0>0$. If $x\in C(R_+, R_+)$ satisfies the following integral inequality containing multiple integrals

$$\begin{aligned}\phi(x(t))\le a(t)&+\int_0^{\alpha_1(t)}f(s,t)\psi(x(s))ds+\int_0^{\alpha_2(t)}g(s,t)\psi(x(s))\omega(x(s))ds\\&+\int_0^{\alpha_3(t)}\int_0^s h(\tau,t)\psi(x(\tau))d\tau ds,\end{aligned}\tag{21}$$

then we can find some $\bar{t}\in R_+$ such that for $t\in[0,\bar{t}]$

$$Y(\overline{H}(t))+\int_0^{\alpha_2(t)}g(s,t)ds\in Dom(Y^{-1}),$$

and

$$x(t)\le\phi^{-1}\{J^{-1}[Y^{-1}(Y(\overline{H}(t))+\int_0^{\alpha_2(t)}g(s,t)ds)]\},\tag{22}$$

where

$$\overline{H}(t)=J(a(t))+\int_0^{\alpha_1(t)}f(s,t)ds+\int_0^{\alpha_3(t)}\int_0^s h(\tau,t)d\tau ds,\tag{23}$$

$$J(t)=\int_{t_0}^{t}\frac{1}{\psi(\phi^{-1}(s))}ds,\ t>t_0, Y(t)=\int_{t_1}^{t}\frac{1}{\omega(\phi^{-1}(J^{-1}(s)))}\ ds,\quad t_1>0,\ t>0.\tag{24}$$

Proof: The proof for the existence of $\bar{t}$ can be referred to Remark 1 in [10]. We notice (22) obviously holds for $t=0$. Now given an arbitrary number $T>0$, $T\in(0,\bar{t}]$. Define

$$\begin{aligned}d(t)=&a(T)+\int_0^{\alpha_1(t)}f(s,T)\psi(x(s))ds+\int_0^{\alpha_2(t)}g(s,T)\psi(x(s))\omega(x(s))ds\\&+\int_0^{\alpha_3(t)}\int_0^s h(\tau,T)\psi(x(\tau))d\tau ds.\end{aligned}$$

Then for $t\in(0, T]$,

$$x(t)\le\phi^{-1}(d(t)),\tag{25}$$

and

$$\begin{aligned}d'(t)&=f(\alpha_1(t),T)\psi(x(\alpha_1(t)))\alpha'_1(t)+g(\alpha_2(t),T)\psi(x(\alpha_2(t)))\omega(x(\alpha_2(t)))\alpha'_2(t)\\&\quad+\alpha'_3(t)\int_0^{\alpha_3(t)}h(\tau,T)\psi(x(\tau))d\tau\\&\le f(\alpha_1(t),T)\psi(\phi^{-1}(d(\alpha_1(t))))\alpha'_1(t)+g(\alpha_2(t),T)\psi(\phi^{-1}(d(\alpha_2(t))))\\&\quad\omega(\phi^{-1}(d(\alpha_2(t))))\alpha'_2(t)+\alpha'_3(t)\psi(\phi^{-1}(d(\alpha_3(t))))\int_0^{\alpha_3(t)}h(\tau,T)d\tau\\&\le f(\alpha_1(t),T)\psi(\phi^{-1}(d(t)))\alpha'_1(t)+g(\alpha_2(t),T)\psi(\phi^{-1}(d(t)))\\&\quad\omega(\phi^{-1}(d(\alpha_2(t))))\alpha'_2(t)+\alpha'_3(t)\psi(\phi^{-1}(d(t)))\int_0^{\alpha_3(t)}h(\tau,T)d\tau.\end{aligned}\tag{26}$$

So

$$\frac{d'(t)}{\psi(\phi^{-1}(d(t)))} \leq f(\alpha_1(t), T)\alpha'_1(t) + g(\alpha_2(t), T)\omega(\phi^{-1}(d(\alpha_2(t))))\alpha'_2(t) \\ + \alpha'_3(t)\int_0^{\alpha_3(t)} h(\tau, T)d\tau. \tag{27}$$

Integrating (27) from 0 to t, considering J is increasing, we can obtain

$$\begin{aligned} d(t) &\leq J^{-1}[J(a(T)) + \int_0^{\alpha_1(t)} f(s, T)ds + \int_0^{\alpha_2(t)} g(s, T)\omega(\phi^{-1}(d(s)))ds \\ &\quad + \int_0^{\alpha_3(t)}\int_0^{s} h(\tau, T)d\tau ds] \\ &\leq J^{-1}[\overline{H}(T) + \int_0^{\alpha_2(t)} g(s, T)\omega(\phi^{-1}(d(s)))ds], \quad t \in (0, T]. \end{aligned} \tag{28}$$

Define $G(t) = \overline{H}(T) + \int_0^{\alpha_2(t)} g(s, T)\omega(\phi^{-1}(d(s)))ds$, then

$$d(t) \leq J^{-1}(G(t)), \quad t \in (0, T], \tag{29}$$

and

$$\begin{aligned} G'(t) &= g(\alpha_2(t), T)\omega(\phi^{-1}(d(\alpha_2(t))))\alpha'_2(t) \\ &\leq g(\alpha_2(t), T)\omega(\phi^{-1}(J^{-1}(G(\alpha_2(t)))))\alpha'_2(t) \\ &\leq g(\alpha_2(t), T)\omega(\phi^{-1}(J^{-1}(G(t)))\alpha'_2(t), \end{aligned} \tag{30}$$

that is,

$$\frac{G'(t)}{\omega(\phi^{-1}(J^{-1}(G(t)))} \leq g(\alpha_2(t), T)\alpha'_2(t). \tag{31}$$

Integrating (31) from 0 to t, considering $G(0) = \overline{H}(T)$ and Y is increasing, it follows

$$G(t) \leq Y^{-1}[Y(\overline{H}(T)) + \textstyle\int_0^{\alpha_2(t)} g(s, T)ds], \quad t \in (0, T]. \tag{32}$$

Combining (25), (29) and (32) we have

$$x(t) \leq \phi^{-1}\{J^{-1}[Y^{-1}(Y(\overline{H}(T)) + \int_0^{\alpha_2(t)} g(s, T)ds)]\}, \quad t \in (0, T]. \tag{33}$$

Taking $t = T$ in (33), it follows

$$x(T) \leq \phi^{-1}\{J^{-1}[Y^{-1}(Y(\overline{H}(T)) + \textstyle\int_0^{\alpha_2(T)} g(s, T)ds)]\}.$$

Considering $T \in (0, \bar{t}]$ is arbitrary, substituting T with t we have completed the proof.

Remark 4: If $h(s, t) \equiv 0$, $\alpha_1(t) = \alpha_2(t) = \alpha(t)$, then Theorem 2.2 becomes Theorem C [10, Theorem 1].

Now we will apply the concept of establishing Theorem 2.2 to the situation with two independent variables.

Theorem 2:3: Assume that $f_i(x, y)$, $g_i(x, y)$, $h_i(x, y) \in C(R_+ \times R_+, R_+)$, $i = 1, 2$, and $\varphi \in C(R_+, R_+)$ is a strictly increasing function with $\lim_{x\to\infty} \phi(x) = \infty$. $a(x, y) \in C(R_+ \times R_+, R_+)$ is nondecreasing in x for every fixed y and nondecreasing in y for every fixed x. $\alpha(x), \beta(y) \in C^1(R_+, R_+)$ are nondecreasing with $\alpha(x) \le x$, $\beta(y) \le y$. $\psi, \omega \in C(R_+, R_+)$ are nondecreasing with $\psi(x) > 0$, $\omega(x) > 0$ for $x \in (0, \infty)$ and $\int_{t_0}^{\infty} \frac{1}{\psi(\phi^{-1}(s))} ds = \infty$, where $0 < t_0 < a(0, 0)$.

If $u \in C(R_+ \times R_+, R_+)$ satisfies the following integral inequality containing multiple integrals

$$\begin{aligned}\phi(u(x,y)) \le a(x,y) &+ \int_0^{\beta(y)} \int_0^{\alpha(x)} [f_1(s,t)\psi(u(s,t)) + g_1(s,t)\psi(u(s,t))\omega(u(s,t)) \\ &+ \int_0^t \int_0^s h_1(\xi,\tau)\psi(u(\xi,\tau))d\xi d\tau] dsdt \\ &+ \int_0^y \int_0^x [f_2(s,t)\psi(u(s,t)) + g_2(s,t)\psi(u(s,t))\omega(u(s,t)) \\ &+ \int_0^t \int_0^s h_2(\xi,\tau)\psi(u(\xi,\tau))d\xi d\tau]\ dsdt,\end{aligned} \tag{34}$$

then we can find some $\bar{x} > 0$, $\bar{y} > 0$ so that for all $x \in [0, \bar{x}]$, $y \in [0, \bar{y}]$

$$Y(\widetilde{H}(x,y)) + \int_0^{\beta(y)} \int_0^{\alpha(x)} g_1(s,t)dsdt + \int_0^y \int_0^x g_2(s,t)dsdt \in Dom(Y^{-1})$$

and

$$\begin{aligned}u(x,y) \le \phi^{-1}\{J^{-1}[Y^{-1}(Y(\widetilde{H}(x,y)) &+ \int_0^{\beta(y)} \int_0^{\alpha(x)} g_1(s,t)\, dsdt \\ &+ \int_0^y \int_0^x g_2(s,t)dsdt)]\}\end{aligned} \tag{35}$$

where J, Y are defined as in Theorem 2.2, and

$$\begin{aligned}\widetilde{H}(x,y) = J(a(x,y)) &+ \int_0^{\beta(y)} \int_0^{\alpha(x)} [f_1(s,t) + \int_0^t \int_0^s h_1(\xi,\tau)d\xi d\tau]\ dsdt \\ &+ \int_0^y \int_0^x [f_2(s,t) + \int_0^t \int_0^s h_2(\xi,\tau)d\xi d\tau]\ dsdt.\end{aligned}$$

Proof: The process for seeking for $\bar{x}, \bar{y}$ can also be referred to Remark 1 in [10].

If we take $x = 0$ or $y = 0$, then (35) holds trivially. Now fix $x_0 \in (0, \bar{x}]$, $y_0 \in (0, \bar{y}]$, and $x \in (0, x_0]$, $y \in (0, y_0]$. Let

$$\begin{aligned}z(x,y) = a(x,y_0) &+ \int_0^{\beta(y)} \int_0^{\alpha(x)} [f_1(s,t)\psi(u(s,t)) + g_1(s,t)\psi(u(s,t))\omega(u(s,t)) \\ &+ \int_0^t \int_0^s h_1(\xi,\tau)\psi(u(\xi,\tau))d\xi d\tau]\ dsdt \\ &+ \int_0^y \int_0^x [f_2(s,t)\psi(u(s,t)) + g_2(s,t)\psi(u(s,t))\omega(u(s,t)) \\ &+ \int_0^t \int_0^s h_2(\xi,\tau)\psi(u(\xi,\tau))d\xi d\tau] dsdt.\end{aligned} \tag{36}$$

Considering $a(x, y)$ is nondecreasing, we have $u(x, y) \le \varphi^{-1}(z(x, y)) \le \varphi^{-1}(z(x_0, y))$. Moreover,

$$
\begin{aligned}
z_y(x_0, y) &= \beta'(y) \int_0^{\alpha(x_0)} [f_1(s,t)\psi(u(s,\beta(y))) + g_1(s,\beta(y))\psi(u(s,\beta(y)))\omega(u(s,\beta(y))) \\
&\quad + \int_0^{\beta(y)} \int_0^s h_1(\xi,\tau)\psi(u(\xi,\tau))d\xi d\tau]\ ds \\
&\quad + \int_0^{x_0} [f_2(s,y)\psi(u(s,y)) + g_2(s,y)\psi(u(s,y))\omega(u(s,y)) \\
&\quad + \int_0^{y} \int_0^s h_2(\xi,\tau)\psi(u(\xi,\tau))d\xi d\tau]ds \\
&\le \{\beta'(y) \int_0^{\alpha(x_0)} [f_1(s,t) + g_1(s,\beta(y))\omega(u(s,\beta(y))) + \int_0^{\beta(y)} \int_0^s h_1(\xi,\tau)d\xi d\tau]\ ds \\
&\quad + \int_0^{x_0} [f_2(s,y)\ \psi(u(s,y)) + g_2(s,y)\omega(u(s,y)) \\
&\quad + \int_0^{y} \int_0^s h_2(\xi,\tau)d\xi d\tau]ds\}\psi(\phi^{-1}(z(x_0,y))).
\end{aligned}
\tag{37}
$$

So

$$
\begin{aligned}
\frac{z_y(x_0,y)}{\psi(\phi^{-1}(z(x_0,y)))} &\le \beta'(y) \int_0^{\alpha(x_0)} [f_1(s,\beta(y)) + g_1(s,\beta(y))\omega(\phi^{-1}(z(s,\beta(y)))) \\
&\quad + \int_0^{\beta(y)} \int_0^s h_1(\xi,\tau)d\xi d\tau]ds + \int_0^{x_0} [f_2(s,y)\ \psi(u(s,y)) + g_2(s,y)\omega(\phi^{-1}(z(s,y))) \\
&\quad + \int_0^{y} \int_0^s h_2(\xi,\tau)d\xi d\tau]ds.
\end{aligned}
\tag{38}
$$

Integrating (38) from 0 to y we have

$$
\begin{aligned}
&J(z(x_0,y)) - J(a(x_0,y_0)) \le \\
&\int_0^{\beta(y)} \int_0^{\alpha(x_0)} [f_1(s,t) + g_1(s,t)\omega(\phi^{-1}(z(s,t))) + \int_0^{t} \int_0^s h_1(\xi,\tau)d\xi d\tau]dsdt \\
&+ \int_0^{y} \int_0^{x_0} [f_2(s,t) + g_2(s,t)\omega(\phi^{-1}(z(s,t))) + \int_0^{t} \int_0^s h_2(\xi,\tau)d\xi d\tau]dsdt.
\end{aligned}
\tag{39}
$$

Let

$$
\begin{aligned}
u(x_0,y) = J(a(x_0,y_0)) &+ \int_0^{\beta(y)} \int_0^{\alpha(x_0)} [f_1(s,t) + g_1(s,t)\omega(\phi^{-1}(z(s,t))) \\
&+ \int_0^{t} \int_0^s h_1(\xi,\tau)d\xi d\tau]\ dsdt + \int_0^{y} \int_0^{x_0} [f_2(s,t) + g_2(s,t)\omega(\phi^{-1}(z(s,t))) \\
&+ \int_0^{t} \int_0^s h_2(\xi,\tau)d\xi d\tau]\ dsdt.
\end{aligned}
\tag{40}
$$

Then

$$
\begin{aligned}
z(x_0,y) &\le J^{-1}[u(x_0,y)] \\
&\le J^{-1}[\widetilde{H}(x_0,y_0) + \int_0^{\beta(y)} \int_0^{\alpha(x_0)} g_1(s,t)\omega(\phi^{-1}(z(s,t)))dsdt \\
&\quad + \int_0^{y} \int_0^{x_0} g_2(s,t)\omega(\phi^{-1}(z(s,t)))dsdt].
\end{aligned}
$$

Furthermore let

$$
\begin{aligned}
v(x_0,y) =& \widetilde{H}(x_0,y_0) + \int_0^{\beta(y)} \int_0^{\alpha(x_0)} g_1(s,t)\omega(\phi^{-1}(z(s,t)))\ dsdt \\
&+ \int_0^{y} \int_0^{x_0} g_2(s,t)\omega(\phi^{-1}(z(s,t)))\ dsdt.
\end{aligned}
$$

Then

$$z(x_0, y) \le J^{-1}[v(x_0, y)], \tag{41}$$

and

$$\begin{aligned} v_y(x_0, y) =& \beta'(y) \int_0^{\alpha(x_0)} g_1(s, \beta(y))\omega(\phi^{-1}(z(s, \beta(y)))) \, ds \\ &+ \int_0^{x_0} g_2(s, y)\omega(\phi^{-1}(z(s, y))) \, ds \\ &\le [\beta'(y) \int_0^{\alpha(x_0)} g_1(s, \beta(y))ds + \int_0^{x_0} g_2(s, y)ds]\omega(\phi^{-1}(J^{-1}(v(x_0, y)))), \end{aligned}$$

that is,

$$\frac{v_y(x_0, y)}{\omega(\phi^{-1}(J^{-1}(v(x_0, y))))} \le \beta'(y) \textstyle\int_0^{\alpha(x_0)} g_1(s, \beta(y))ds + \int_0^{x_0} g_2(s, y)ds. \tag{42}$$

Integrating (42) from 0 to y, considering $v(x_0, 0) = \widetilde{H}(x_0, y_0)$ we have

$$Y(v(x_0, y)) - Y(\widetilde{H}(x_0, y_0)) \le \textstyle\int_0^{\beta(y)} \int_0^{\alpha(x_0)} g_1(s, t)dsdt + \int_0^{y} \int_0^{x_0} g_2(s, t) \, dsdt.e^{i\theta}$$

Then

$$v(x_0, y) \le Y^{-1}[Y(\widetilde{H}(x_0, y_0)) + \textstyle\int_0^{\beta(y)} \int_0^{\alpha(x_0)} g_1(s, t)dsdt + \int_0^{y} \int_0^{x_0} g_2(s, t) \, dsdt],$$

and

$$\begin{aligned} u(x, y) \le \phi^{-1}[J^{-1}(v(x_0, y))] \le \phi^{-1}\{J^{-1}[Y^{-1}(Y(\widetilde{H}(x_0, y_0)) \\ + \int_0^{\beta(y)} \int_0^{\alpha(x_0)} g_1(s, t)dsdt + \int_0^{y} \int_0^{x_0} g_2(s, t)dsdt)]\}. \end{aligned} \tag{43}$$

Take $x = x_0$, $y = y_0$ and we have

$$\begin{aligned} u(x_0, y_0) \le& \phi^{-1}\{J^{-1}[Y^{-1}(Y(\widetilde{H}(x_0, y_0)) + \int_0^{\beta(y_0)} \int_0^{\alpha(x_0)} g_1(s, t)dsdt \\ &+ \int_0^{y_0} \int_0^{x_0} g_2(s, t)dsdt)]\}. \end{aligned} \tag{44}$$

Since $x_0 \in (0, \overline{x}]$, $y_0 \in (0, \overline{y}]$ are arbitrary, substitute x_0, y_0 with x, y and the proof is complete.

Corollary 2.4: Assume that $f(x, y)$, $g(x, y)$, $h(x, y) \in C(R_+ \times R_+, R_+)$, and a, φ, ψ, ω, α, β, J, Y are defined as in Theorem 2.3. If $u \in C(R_+ \times R_+, R_+)$ satisfies the following integral inequality containing multiple integrals

$$\begin{aligned} \phi(u(x, y)) \le& a(x, y) + \int_0^{\beta(y)} \int_0^{\alpha(x)} [f(s, t)\psi(u(s, t)) + g(s, t)\psi(u(s, t))\omega(u(s, t)) \\ &+ \int_0^{t} \int_0^{s} h(\xi, \tau)\psi(u(\xi, \tau))d\xi d\tau] \, dsdt, \end{aligned}$$

then we can find some $\overline{x} > 0$, $\overline{y} > 0$ such that for all $x \in [0, \overline{x}]$, $y \in [0, \overline{y}]$

$$Y(\widetilde{H}(x, y)) + \int_0^{\beta(y)} \int_0^{\alpha(x)} g_1(s, t) \, dsdt \in Dom(Y^{-1}),$$

and

$$u(x,y) \le \phi^{-1}\{J^{-1}[Y^{-1}(Y(\widetilde{H}(x,y)) + \int_0^{\beta(y)}\int_0^{\alpha(x)} g(s,t)dsdt)]\},$$

where

$$\widetilde{H}(x,y) = J(a(x,y)) + \int_0^{\beta(y)}\int_0^{\alpha(x)} [f(s,t) + \int_0^t \int_0^s h(\xi,\tau)d\xi d\tau]\, dsdt].$$

Remark 5: If we take $h(x, y) \equiv 0$, $\psi\ (u(x, y)) = u^n(x, y)$, $\varphi(x, y) = u^m(x, y)$, $m > n > 0$, then Corollary 2.4 reduces to Theorem D [11, Theorem 2.1].

Corollary 2.5: Assume that f_i, $g_i(x, y) \in C(R_+ \times R_+, R_+)$, $i = 1, 2$, and a, φ, ψ, ω, J, Y are defined as in Theorem 2.3. If $u \in C(R_+ \times R_+, R_+)$ satisfies the following integral inequality containing multiple integrals

$$\phi(u(x,y)) \le a(x,y) + \int_0^{\beta(y)}\int_0^{\alpha(x)} [f_1(s,t)\psi(u(s,t)) + g_1(s,t)\psi(u(s,t))\omega(u(s,t))]dsdt$$
$$+ \int_0^{y}\int_0^{x} [f_2(s,t)\psi(u(s,t)) + g_2(s,t)\psi(u(s,t))\omega(u(s,t))]dsdt,$$

then we can find some $\overline{x} > 0$, $\overline{y} > 0$ such that for all $x \in [0, \overline{x}]$, $y \in [0, \overline{y}]$

$$Y(\widetilde{H}(x,y)) + \int_0^{\beta(y)}\int_0^{\alpha(x)} g_1(s,t)\, dsdt + \int_0^{y}\int_0^{x} g_2(s,t)\, dsdt \in Dom(Y^{-1}),$$

and

$$u(x,y) \le \phi^{-1}\{J^{-1}[Y^{-1}(Y(\widetilde{H}(x,y)) + \int_0^{\beta(y)}\int_0^{\alpha(x)} g_1(s,t)\, dsdt$$
$$+ \int_0^{y}\int_0^{x} g_2(s,t)\, dsdt)]\},$$

where

$$\widetilde{H}(x,y) = J(a(x,y)) + \int_0^{\beta(y)}\int_0^{\alpha(x)} f_1(s,t)\, dsdt + \int_0^{y}\int_0^{x} f_2(s,t)\, dsdt.$$

Remark 6: If we take $f_1(x, y) = f_2(x, y) \equiv 0$, $\psi(u(x, y)) = u^n(x, y)$, $\varphi(x, y) = u^m(x, y)$, $m > n$, then Corollary 8 reduces to Theorem E [11, Theorem 2.2].

3 Applications

In this section, we will present two examples in order to illustrate the validity of the above results. In the first example, we will try to prove the global existence of the solutions of a delay differential equation, while in the second example, we will obtain the bound of the solutions of an integral equation.

For the sake of proving the global existence of solutions of differential equations, we first recall some basic facts. Consider the following equation

$$\begin{cases} X'(t) = H(t, X(t), X(\alpha(t))) \\ X(0) = X_0 \end{cases} \tag{45}$$

with $X_0 \in R^n$, $H \in C(R_+ \times R^{2n}, R^n)$, $\alpha \in C^1(R_+, R_+)$ satisfying $\alpha(t) \le t$ for $t \ge 0$. A result in [13] guarantees that for every $X_0 \in R^n$, Equation 45 has a solution, but the uniqueness of solutions cannot be guaranteed. Furthermore, every solution of (45) has a maximal time of existence $T > 0$, and if $T < \infty$, then $\lim_{t\to\infty} sup \parallel X(t) \parallel = \infty$.

Example 1: Consider the following differential equation group

$$\begin{cases} (x^p(t))' = y^{\frac{p+q}{2}}(t) + F(x(t), t) \\ (y^p(t))' = G(t, x(\alpha(t))) \end{cases} \tag{46}$$

where p is an even number. $\alpha(t)$ is a nondecreasing function, $\alpha(t) \in C^1(R_+, R_+)$, $\alpha(t) \le t$, $\forall t \ge 0$. $p > q > 0$. Assume $|F(x(t), t)| \le \widetilde{f}_1(t)|x^q(t)|v(|x|)$, $\widetilde{f}_3(t) = max(1, \tilde{f}_1(t))$, $\widetilde{f}_3(t) = max(1, \tilde{f}_1(t))$, where $\widetilde{f}_1, \widetilde{f}_2, v \in C(R_+, R_+)$, $\int_0^\infty \frac{1}{v(s)} ds = \infty$, and v is nondecreasing.

Let $u^p(t) = x^p(t) + y^p(t)$, then

$$(u^p(t))' = (x^p(t))' + (y^p(t))' = y^{\frac{p+q}{2}}(t) + F(x(t), t) + G(t, x(\alpha(t))) \tag{47}$$

If $(x(t), y(t))$ is a solution of (46) defined on the maximal existence interval $[0, T)$, integrating (47) from 0 to t, we have

$$\begin{aligned} |u^p(t)| &\le |u^p(0)| + \int_0^t [|y^{\frac{p+q}{2}}(s)| + |F(x(s), s)| + |G(s, x(\alpha(s)))|]\, ds \\ &\le \int_0^t [|u^{\frac{p+q}{2}}(s)| + \widetilde{f}_1(s)|x^q(s)|v(|x|) + \widetilde{f}_2(s)|x^q(\alpha(s))|]\, ds \\ &\le \int_0^t [|u^{\frac{p+q}{2}}(s)| + \widetilde{f}(s)|u^q(s)|v(|u|) + \widetilde{f}_2(s)|u^q(\alpha(s))|]\, ds. \end{aligned}$$

Then

$$\begin{aligned} |u^p(t)| &\le |u^p(0)| + \int_0^t [|u^{\frac{p+q}{2}}(s)| + \widetilde{f}_1(s)|u^q(s)|v(|u|) + \widetilde{f}_2(s)|u^q(\alpha(s))|]\, ds \\ &\le |u^p(0)| + \int_0^t [\widetilde{f}_3(s)|u^q(s)|\omega(|u|) + \widetilde{f}_2(s)|u^q(\alpha(s))|]\, ds \\ &= |u^p(0)| + \int_0^t \widetilde{f}_3(s)|u^q(s)|\omega(|u|) + \int_0^{\alpha(t)} \frac{\widetilde{f}_2(\alpha^{-1}r)}{\alpha'(\alpha^{-1}r)}|u^q(r)|dr, \end{aligned}$$

where $\omega(|u| = v(|u|) + |u^{\frac{p-q}{2}}|$. From Theorem 2.1 we have

$$\begin{aligned} |u(t)| &\le \{\Omega^{-1}[\Omega(|u^{p-q}(0)| + \frac{p-q}{p}\int_0^{\alpha(t)} \frac{\tilde{f}_2(\alpha^{-1}r)}{\alpha'(\alpha^{-1}r)}dr) + \frac{p-q}{p}\int_0^t \tilde{f}_3(s)ds]\}^{\frac{1}{p-q}} \\ &= \{\Omega^{-1}[\Omega(|u^{p-q}(0)| + \frac{p-q}{p}\int_0^t \tilde{f}_2(s)ds) + \frac{p-q}{p}\int_0^t \tilde{f}_3(s)ds]\}^{\frac{1}{p-q}}, 0 \le t < T. \end{aligned}$$

Obviously we have $\{|x(t)|, |y(t)|\} \le |u(t)|$. So $x(t)$, $y(t)$ do not blow up in finite time. Then $T = \infty$, and the solutions of (46) are global.

Example 2: Considering the following integral equation

$$u(x,y)ln(u(x,y)+1) = a(x,y) + \int_0^{\beta(y)} \int_0^{\alpha(x)} [F(s,t,u(x,y)) + G(s,t,u(x,y))]\, dsdt, \quad (48)$$

where $u \in C(R_+ \times R_+, R_+)$, $|F(x, y, u(x, y))| \leq f(x, y)u(x, y)$,
$|G(x, y, u(x, y))| \leq g(x, y)u^2(x, y)$, $f, g \in C(R_+ \times R_+, R_+)$, $a(x, y)$, $\alpha(x)$, $\beta(y)$ are defined as in Theorem 2.3.

Let $\varphi(u) = uln(u + 1)$, $\omega(u) = u$, $\eta(u) = u$. Then one can easily see the conditions of Theorem 2.3 are satisfied. So we can obtain the bound of $u(x, y)$ as

$$u(x,y) \leq \phi^{-1}\{J^{-1}[\Upsilon^{-1}(\Upsilon(\widetilde{H}(x,y)) + \int_0^{\beta(y)} \int_0^{\alpha(x)} g(s,t)dsdt]\}, \quad x \in [0,\bar{x}], \quad y \in [0,\bar{y}], \quad (49)$$

where $\bar{x}$, $\bar{y}$ are determined similar to the process in Theorem 2.3, and

$$\widetilde{H}(x,y) = J(a(x,y)) + \int_0^{\beta(y)} \int_0^{\alpha(x)} f(s,t)\, dsdt.$$

Remark 7: we note that the methods in [1-12] are not available for the estimate of bound for the solution of Equation 48.

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5 Authors' contributions
BZ carried out the main part of this article. Both of the two authors read and approved the final manuscript.

4 Competing interests
The authors declare that they have no competing interests.

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